

Lagrange–Ricci Flows and Evolution of Geometric Mechanics & Analogous Gravity on Lie Algebroids

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Abstract

The approach to nonholonomic Ricci flows and geometric evolution of regular Lagrange systems [S. Vacaru: J. Math. Phys. **49** (2008) 043504 & Rep. Math. Phys. **63** (2009) 95] is extended to include geometric mechanics and gravity models on Lie algebroids. We prove that such evolution scenarios of geometric mechanics and analogous gravity can be modeled as gradient flows characterized by generalized Perelman functionals if an equivalent geometrization of Lagrange mechanics [J. Kern, Arch. Math. (Basel) **25** (1974) 438] is considered. The R. Hamilton equations on Lie algebroids describing Lagrange–Ricci flows are derived. Finally, we show that geometric evolution models on Lie algebroids is described by effective thermodynamical values derived from statistical functionals on prolongation Lie algebroids.

Keywords: Ricci flows, Lie algebroids, Lagrange mechanics, analogous gravity.

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1 Introduction

The Ricci flow theory [1, 2] became attractive for research in mathematics and physics after G. Perelman successfully carried out his program [3, 4, 5] which resulted in proofs of Thurston and Poincaré conjectures, see

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reviews of results in Refs. [6, 7, 8]. The profound impact of such results on understanding the topology and geometric structure of curved spacetime and fundamental properties of classical and quantum interactions was used as a motivation to study the geometric evolution of regular Lagrange systems [9, 10] on tangent bundles and nonholonomic (pseudo) Riemannian and Einstein manifolds. We developed Ricci flow theories for classical and quantum solutions of Einstein equations, generalizations to noncommutative, Finsler, diffusion, fractional spaces etc, see [11, 12] and references therein.

Effective Lagrange and Hamilton models, Lie algebroid and almost Kähler and Dirac structures are considered, for instance, in quantum gravity and modified gravity theories when Ricci flows on parameters are derived from a renormalization procedure with running/evolution of physical parameters etc [13, 14, 15]. One of the important tasks in modern geometry and physics is to elaborate and analyze flow evolution of more complex geometries and physical systems with nontrivial topology, generalized symmetries, nonholonomic constraints etc. So, it is not an academic exercise and "pure" geometric interest to perform generalizations of the Ricci flow theory derived for Lagrangians/Hamiltonians on Lie algebroids. Fundamental properties of spacetime topology seem to be related to a series of important questions on dimensions of real mechanical systems and physical interactions, analogous gravity modelling, renormalizability of certain quantum theories, possible modifications of gravity derived from modern cosmology observations etc. We need rigorous studies on evolution of theories with rich geometric structure, generalized and deformed symmetries and symplectic structure and nonholonomic constraints.

Specifically, the goal of this paper is to elaborate a model of geometric evolution of Lagrange mechanics and analogous gravity theory on Lie algebroids using certain constructions proposed and developed in Refs. [16, 17, 18, 19, 20]. The key idea considered in our works is that physical theories can be encoded into the geometry of generalized nonholonomic spaces (defined by corresponding classes of non-integrable constraints on fundamental dynamical and evolution equations) via "standard", or analogous, geometric objects like metrics, (almost) symplectic forms, nonlinear and linear connections, related curvatures and torsions, and their geometric flows evolution. A subclass of evolution scenarios are uniquely determined following geometric principles for entropy type functionals derived for families of generating Lagrange functions $L(\mathbf{x}, \mathbf{y}, \chi)$.¹ Hopefully, such assumptions on geometric

¹We treat (\mathbf{x}, \mathbf{y}) as some generalized coordinates (for instance, on a tangent bundle TM , or a Lie algebroid E over a manifold M) and χ is a real evolution parameter. In

evolution mechanics allows us to formulate an alternative and very different approach to and provide us new possibilities to explore properties of Lagrange systems using methods in geometric analysis.

Theories of Lagrange and Hamilton systems on Lie algebroids (and various discrete analogs on Lie groupoids, Poisson structures and algebroids etc) were proposed [21, 22] and actively developed during last ten years, see original contributions and reviews of results in Refs. [13, 23, 24, 25]. On Lie algebroid gravity and gauge interactions models, we cite [26, 18, 19] and references therein. The inclusive nature of Lie algebroid formalism allows us to describe very different situations in mechanics and physics such as Lagrangian systems with symmetry and nonholonomic constraints, theories with semidirect products and/or evolving on Lie algebras and generalizations. It is possible in such cases to derive some Lagrange/Euler–Poincaré, or Euler–Lagrange equations and geometrize such systems as generalized Poisson geometries etc. New tools have been introduced and new understanding have been provided, for instance, by the multi-symplectic formalism and Poisson–Nijenhuis Lie algebroid theory etc.

Nevertheless, we have to consider additional and alternative constructions for above mentioned algebroid models of geometric mechanics and classical/quantum field theories if we want to study the Ricci flow evolution of systems and spaces with "rich" geometric and physical structure keeping certain analogy with the Hamilton–Perelman theory. It is not clear how the standard formalism elaborated for Ricci flows of (semi) Riemann and (almost) Kähler geometries can be extended to describe directly flow evolution of models of Lie algebroid mechanics developed in Refs. [21, 22, 23, 24, 25].

Our proposal is to use J. Kern’s [16] constructions on Lagrange spaces (the term is due to that article developing in a "nonhomogenous" manner the M. Matsumoto results on Finsler connections [17], see references therein; further developments and applications, for instance, in modern classical and quantum gravity [11, 14, 15]. In such an approach, the nondegenerate Hessian of a regular Lagrangian can be treated as a metric structure for fibers on TM which can be extended on total space using the so-called Sasaki lifts [28]. It is involved also a corresponding semi-spray structure inducing a canonical nonlinear connection (in brief, N-connection; the global definition is due to [27], see historical remarks and applications in modern mechanics and gravity in [14]). For such geometric data, a model of Lagrange–Ricci

certain "dual" and, in some sense, more general approaches, we can consider families of Hamiltonians $H(\mathbf{x}, \mathbf{p}, \chi)$, (almost) symplectic and/or Poisson structures with associated co-tangent bundle T^*M etc. Here we also note that in our works there are used left low/up indices as labels for some geometric objects and/or spaces.

flow theory [9, 10] can be formulated in N-adapted form, via corresponding generalizations of Perelman's functionals, on tangent bundles and/or nonholonomic² (semi) Riemannian manifolds.

The paper is organized as follows. In section 2, we survey the geometry of Lie algebroids and prolongations and geometrization of Lagrange mechanics on such spaces following approach from [13, 23, 24, 25]. We summarize necessary tools from the geometry of N-connections on prolongation Lie algebroids in section 3. The constructions are performed in metric compatible form which allows us to formulate an analogous N-adapted gravity model on Lie algebroids. An alternative geometrization of regular Lagrange mechanics and analogous modeling of gravity following Kern–Matsumoto ideas extended for prolongation Lie algebroids is provided. Section 4 is devoted to Main Theorems for Lagrange–Ricci flows on prolongation Lie algebroids.

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2 Lagrange Mechanics and Lie Algebroids

We outline basic concepts and definitions for Lie algebroids and geometric mechanics with regular Lagrangians on prolongations of Lie algebroids on bundle maps, see in Refs. [13, 23, 25] and references therein.

2.1 Linear connections and metrics on Lie algebroids

A **Lie algebroid** $\mathcal{E} = (E, [\cdot, \cdot], \rho)$ over a manifold M is a triple defined by 1) a real *vector bundle* $\tau : E \rightarrow M$ together with 2) a *Lie bracket* $[\cdot, \cdot]$ on the spaces of global sections $Sec(\tau)$ of map τ and 3) the *anchor* map $\rho : E \rightarrow TM$ defined as a bundle map over identity and constructed such that the homomorphism $\rho : Sec(\tau) \rightarrow \mathcal{X}(M)$ of $C^\infty(M)$ -modules \mathcal{X} induced this map satisfies the condition

$$[X, fY] = f[X, Y] + \rho(X)(f)Y, \quad \forall X, Y \in Sec(\tau) \text{ and } f \in C^\infty(M).$$

For a Lie algebroid, the anchor map ρ is equivalent to a homomorphism between the Lie algebras $(Sec(\tau), [\cdot, \cdot])$ and $(\mathcal{X}(M), [\cdot, \cdot])$.

In local form, the properties of a Lie algebroid \mathcal{E} are determined by the local functions $\rho_\alpha^i(x^k)$ and $C_{\alpha\beta}^\gamma(x^k)$ on M , where $x = \{x^k\}$ are local coordinates on a chart $U \subset M$, with $\rho(e_\alpha) = \rho_\alpha^i(x)\partial_i$ and $[e_\alpha, e_\beta] = C_{\alpha\beta}^\gamma(x)e_\gamma$,

²equivalently, anholonomic and/or non-integrable, which refer to certain classes of distributions on a geometric/physical space

satisfying the following equations

$$\rho_\alpha^i \partial_i \rho_\beta^j - \rho_\beta^i \partial_i \rho_\alpha^j = \rho_\gamma^j C_{\alpha\beta}^\gamma \quad \text{and} \quad \sum_{\text{cyclic } (\alpha, \beta, \gamma)} \left(\rho_\alpha^i \partial_i C_{\beta\gamma}^\nu + C_{\beta\gamma}^\mu C_{\alpha\mu}^\nu \right) = 0.$$

A linear connection D on \mathcal{E} is defined as a \mathbb{R} -bilinear map $D : Sec(E) \times Sec(E) \rightarrow Sec(E)$ such that $\forall f \in C^\infty(M)$ and $\forall X, Y \in Sec(E)$ this covariant derivative operator satisfies the conditions

$$D_f X Y = f D_X Y \quad \text{and} \quad D_X(fY) = \rho(X)(f)Y + f D_X Y. \quad (1)$$

Locally, D is given by its coefficients $\Gamma_{\beta\gamma}^\alpha$ when $D_X Y = X^\alpha (\rho_\alpha^i \partial_i Y^\gamma + \Gamma_{\alpha\beta}^\gamma Y^\beta) e_\gamma$, where $X = X^\alpha e_\alpha$ and $Y = Y^\alpha e_\alpha$ for a local basis $\{e_\alpha\} \in Sec(E)$. A curve $a : I \rightarrow E$ given by a function $a(\tau) = a^\alpha(\tau) e_\alpha$ on a real parameter τ , is said to be an auto-parallel of D if $D_a a = 0$.

The exterior differential on \mathcal{E} can be defined in standard form using the operator d on \mathcal{E} , when $d : Sec(\bigwedge^k \tau^*) \rightarrow Sec(\bigwedge^{k+1} \tau^*)$, $d^2 = 0$, where \bigwedge is the antisymmetric product operator, see details in Refs. [29, 23, 25, 13]. The local contributions of a N-connection can be seen from such formulas for a smooth formula $f : M \rightarrow \mathbb{R}$, $df(X) = \rho(X)f$, for $X \in Sec(\tau)$, when

$$dx^i = \rho_\alpha^i e^\alpha \quad \text{and} \quad de^\gamma = -\frac{1}{2} C_{\alpha\beta}^\gamma e^\alpha \wedge e^\beta.$$

With respect to any section X , we can define the Lie derivative

$$\mathcal{L}_X = i_X \circ d + d \circ i_X : Sec(\bigwedge^k \tau^*) \rightarrow Sec(\bigwedge^k \tau^*), \quad (2)$$

using the cohomology operator d and its inverse i_X , see details in [30, 25, 13].

A metric ϖ on \mathcal{E} is defined as a map

$$\varpi : E \times_M E \rightarrow \mathbb{R}. \quad (3)$$

Locally, $\varpi = \varpi_{\alpha\beta}(x) e^\alpha \otimes e^\beta$. We shall use also the inverse matrix/ metric, $\varpi^{\alpha\beta}(x)$.

There is a "preferred" linear connection $\varpi \nabla$ on \mathcal{E} (the analog of the Levi-Civita connection in Riemannian geometry) completely defined by a metric ϖ . This connection is uniquely determined following two conditions:

$$\begin{aligned} \varpi T(X, Y) &= \varpi \nabla_X Y - \varpi \nabla_Y X + [X, Y] = 0, \text{ i.e. zero torsion;} \\ \varpi(\varpi \nabla_X Y, Z) + \varpi(Y, \varpi \nabla_X Z) - \rho(X)(\varpi(Y, Z)) &= 0, \text{ i.e. metricity,} \end{aligned}$$

which result in formula

$$\begin{aligned} 2\varpi(\varpi\nabla_X Y, Z) &= \rho(X)(\varpi(Y, Z)) + \rho(Y)(\varpi(X, Z)) - \rho(Z)(\varpi(X, Y)) \\ &\quad - \varpi(X, [Y, Z]) + \varpi(Y, [Z, X]) - \varpi(Z, [Y, X]), \end{aligned}$$

$\forall X, Y, Z \in \text{Sec}(E)$. The curvature of $\varpi\nabla$ on \mathcal{E} (the analog of the Riemannian tensor on standard manifolds) is defined in standard form

$$\varpi R(X, Y)Z = (\varpi\nabla_X \varpi\nabla_Y - \varpi\nabla_Y \varpi\nabla_X - \varpi\nabla_{[X, Y]})Z.$$

Introducing in above formulas $X = e_\alpha, Y = e_\beta, Z = e_\gamma$, for

$$\begin{aligned} \varpi\nabla_{e_\alpha} e_\beta &:= \varpi\nabla_\alpha e_\beta = \Gamma_{\beta\alpha}^\gamma e_\gamma, \\ \Gamma_{\beta\gamma}^\alpha &= \frac{1}{2}\varpi^{\alpha\varphi}(\rho_\gamma^i \partial_i \varpi_{\beta\varphi} + \rho_\beta^i \partial_i \varpi_{\gamma\varphi} - \rho_\varphi^i \partial_i \varpi_{\beta\gamma} \\ &\quad + C_{\varphi\gamma}^\tau \varpi_{\tau\beta} + C_{\varphi\beta}^\tau \varpi_{\tau\gamma} - C_{\beta\gamma}^\tau \varpi_{\tau\varphi}), \end{aligned} \tag{4}$$

we compute the coefficients of torsion and curvature of the Levi-Civita connection, respectively,

$$\begin{aligned} \varpi T_{\beta\alpha}^\gamma &= \Gamma_{\beta\alpha}^\gamma - \Gamma_{\alpha\beta}^\gamma + C_{\alpha\beta}^\gamma = 0 \text{ and} \\ \varpi R_{\beta\gamma\delta}^\alpha &= \rho_\delta^i \partial_i \Gamma_{\beta\gamma}^\alpha - \rho_\gamma^i \partial_i \Gamma_{\beta\delta}^\alpha + \Gamma_{\beta\gamma}^\varphi \Gamma_{\varphi\delta}^\alpha - \Gamma_{\beta\delta}^\varphi \Gamma_{\varphi\gamma}^\alpha + \Gamma_{\beta\varphi}^\alpha C_{\gamma\delta}^\varphi. \end{aligned}$$

In standard form, we define the Ricci tensor contracting respective indices, $\varpi Ric = \{\varpi R_{\beta\gamma} := \varpi R_{\beta\gamma\alpha}^\alpha\}$, and the scalar curvature, $\varpi R := \varpi^{\alpha\beta} \varpi R_{\alpha\beta}$. Such formulas are very similar to those for (pseudo) Riemannian geometry formulated in nonholonomic bases satisfying anholonomy relations for some nontrivial coefficients $C_{\gamma\delta}^\varphi$. For the case of Lie algebroids, the fundamental geometric objects on the space $\text{Sec}(E)$. The above formulas on metrics, connections and Ricci tensors can be used for elaborating a Ricci Lie algebroid evolution theory. Nevertheless, there are necessary a number of additional assumptions and constructions in order to include in such a scheme models of Lagrange mechanics and classical and quantum field theories.

2.2 The prolongation of Lie algebroids and Lagrange mechanics

In Refs. [23, 25, 13], a geometric formalism for Lagrange mechanics on Lie algebroids was developed using the concept of prolongation of a Lie algebroid over a fibration (in brief, prolongation Lie algebroid). Let us briefly outline some basic constructions.

Consider a Lie algebroid $\mathcal{E} = (E, [\cdot, \cdot], \rho)$ and a fibration $\pi : P \rightarrow M$ both defined over the same manifold M . We denote local coordinates in the form $(x^i, u^A) \in P$ write $\{e_\alpha\}$ for a local basis of sections of E . For our purposes, we can consider that $P = E$. The anchor map $\rho : E \rightarrow TM$ and the tangent map $T\pi : TP \rightarrow TM$, can be used to construct a subset $\mathcal{T}_s^E P := \{(b, v) \in E_x \times T_x P; \rho(b) = T_p \pi(v); p \in P_x, \pi(p) = x \in M\}$. Globalizing the construction, we obtain another Lie algebroid, $\mathcal{T}^E P := \bigcup_{s \in S} \mathcal{T}_s^E P$, which is called the prolongation of E over π . Equivalently, $\mathcal{T}^E P$ is called the E -tangent bundle to π , which is also a vector bundle over P , with projection τ_P^E just onto the first factor, $\tau_P^E(b, v) = b$. The elements of $\mathcal{T}^E P$ are written (p, b, v) . There are also used brief denotations $(p, b, v) \in \mathcal{T}_p^E P \rightarrow (b, v) \in \mathcal{T}^E P$ if not ambiguities. The anchor $\rho^\pi : \mathcal{T}^E P \rightarrow TP$ is given by maps $\rho^\pi(p, b, v) = v$, i.e. projection onto the third factor.

It is possible to define also the projection onto the second factor (i.e. a morphism of Lie algebroids over π), $\mathcal{T}\pi : \mathcal{T}^E P \rightarrow E$, when $\mathcal{T}\pi(p, b, v) = b$. For instance, an element $(p_1, b_1, v_1) \in \mathcal{T}^E P$ is vertical if $\mathcal{T}\pi(p_1, b_1, v_1) = b_1 = 0$, i.e. such elements are of type $(p, 0, v)$ when v is a π -vertical vector (tangent to P at point p).

Locally, any element $\bar{z} = (p, b, v) \in \mathcal{T}^E P$, when $b = z^\alpha e_\alpha$ and $v = \rho_\alpha^i z^\alpha \partial_i + v^A \partial_A$, for $\partial/\partial u^A$, can be decomposed $\bar{z} = z^\alpha \mathcal{X}_\alpha + v^A \mathcal{V}_A$, where $(\mathcal{X}_\alpha, \mathcal{V}_A)$, with vertical \mathcal{V}_A , define a local basis of sections of $\mathcal{T}^E P$. In explicit form, such bases can be parametrized in the form $\mathcal{X}_\alpha = \mathcal{X}_\alpha(p) = (e_\alpha(\pi(p)), \rho_\alpha^i \partial_{i|p})$ and $\mathcal{V}_A = (0, \partial_{A|p})$ where partial derivatives are taken in a point $p \in S_x$.

The Lie algebroid structure of $\mathcal{T}^E P$ is stated by the anchor map $\rho^\pi(Z) = \rho_\alpha^i Z^\alpha \partial_i + V^A \partial_A$ acting on sections Z with associated decompositions of type \bar{z} and by the Lie brackets $[\mathcal{X}_\alpha, \mathcal{X}_\beta]^\pi = C_{\alpha\beta}^\gamma \mathcal{X}_\gamma$, $[\mathcal{X}_\alpha, \mathcal{V}_B]^\pi = 0$, $[\mathcal{V}_A, \mathcal{V}_B]^\pi = 0$. Using dual bases $(\mathcal{X}^\alpha, \mathcal{V}^B)$, we can perform an exterior differential calculus following formulas

$$dx^i = \rho_\alpha^i \mathcal{X}^\alpha, \text{ for } d\mathcal{X}^\gamma = -\frac{1}{2} C_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \text{ and } du^A = \mathcal{V}^A, \text{ for } d\mathcal{V}^A = 0. \quad (5)$$

For instance, if we take a (function, or Lagrangian) $L(x^i, u^\alpha)$ on E we can compute

$$d^E L = \rho_\alpha^i (\partial_i L) \mathcal{X}^\alpha + (\partial_A L) \mathcal{V}^A,$$

where $d^E x^i = \rho_\alpha^i \mathcal{X}^\alpha$ and $d^E u^\alpha = \mathcal{V}^\alpha$. We shall write the absolute differential, for instance, of L , in the form $d^E L$ for $dx^i \rightarrow d^E x^i$ and $du^\alpha \rightarrow d^E u^\alpha$ if $P = E$ and $\mathcal{V}^A \rightarrow \mathcal{V}^\alpha$.

Let us consider $P = E$ for $\mathcal{T}^E P$ when the prolongation Lie algebroid is for a bundle projection $\tau : E \rightarrow M$, we can formulate a mechanical model for a Lagrangian function $L \in C^\infty(E)$ and chose a vertical endomorphism $S : \mathcal{T}^E E \rightarrow \mathcal{T}^E E$ of type $S(a, b, v) = \xi^V(a, b) = (a, 0, b_a^V)$, where b_a^V is the vector tangent to the curve $a + \tau b$ when the parameter $\tau = 0$. The vertical lift ξ^V allows us to define a map $\xi^V : \tau^* E \rightarrow \mathcal{T}^E E$ and the Liouville dilaton vector field $\Delta(a) = \xi^V(a, a) = (a, 0, b_a^V)$.

A model of Lie algebroid mechanics for a Lagrangian L can be geometrized on $\mathcal{T}^E E$ in terms of three geometric objects,

$$\begin{aligned} \text{the Cartan 1-section: } \theta_L &:= S^*(dL) \in \text{Sec}((\mathcal{T}^E E)^*); \\ \text{the Cartan 2-section: } \omega_L &:= -d\theta_L \in \text{Sec}(\wedge^2(\mathcal{T}^E E)^*); \\ \text{the Lagrangian energy: } E_L &:= \mathcal{L}_\Delta L - L \in C^\infty(E), \end{aligned} \quad (6)$$

where the Lie derivative (2) is considered in the last formula. Using these variables, the dynamical equations derived for L can be geometrized as

$$i_{SX}\omega_L = -S^*(i_X\omega_L) \quad \text{and} \quad i_\Delta\omega_L = -S^*(dE_L), \forall X \in \text{Sec}(\mathcal{T}^E E). \quad (7)$$

Such geometric equations define equivalently a regular Lagrange mechanics if ω_L is regular at every point as a bi-linear form, i.e. it is a symplectic section. For configurations with regular L , and ω_L , there exists a unique solution Γ_L and a form Ω_L satisfying the condition $i_{\Gamma_L}\Omega_L = dE_L$. From equations (7), we obtain $i_{S\Gamma_L}\omega_L = i_\Delta\omega_L$. This states that $S(\Gamma_L) = \Delta$ (equivalently, $\mathcal{T}\tau(\Gamma_L(a)) = a, \forall a \in E$) which constraints Γ_L to be a SODE (second order differential equation) section, or semispray. Taking $\Omega_L = \omega_L$, for $P = E$, we can write the last equation as a symplectic equation

$$i_{\Gamma_L}\omega_L = d^E E_L, \quad (8)$$

for $\Gamma_L \in \text{Sec}(\mathcal{T}^E E)$.

The above geometric objects (6) and equations (7) can be written in coefficient forms. Introducing local coordinates $(x^i, y^\alpha) \in E$, for Lie algebroid structure functions $(\rho_\alpha^i, C_{\alpha\beta}^\gamma)$, and choosing a basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\} \in \text{Sec}(\mathcal{T}^E E)$, for all α , we have

$$\begin{aligned} S\mathcal{X}_\alpha &= \mathcal{V}_\alpha, \quad S\mathcal{V}_\alpha = 0, \quad \Delta = y^\alpha \mathcal{V}_\alpha, \quad E_L = y^\alpha \partial L / \partial y^\alpha - L, \\ \omega_L &= \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \mathcal{X}^\alpha \wedge \mathcal{V}^\beta + \frac{1}{2} (\rho_\beta^i \frac{\partial^2 L}{\partial x^i \partial y^\alpha} - \rho_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} + C_{\alpha\beta}^\gamma \frac{\partial L}{\partial y^\gamma}) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta. \end{aligned} \quad (9)$$

As a vertical endomorphism (equivalently, tangent structure) can be used the operator $S := \mathcal{X}^\alpha \otimes \mathcal{V}_\alpha$.

The Euler–Lagrange section associated with L is given by $\Gamma_L = y^\alpha \mathcal{X}_\alpha + \varphi^\alpha \mathcal{V}_\alpha$, when functions $\varphi^\alpha(x^i, y^\beta)$ solve this system of linear equations

$$\varphi^\beta \frac{\partial^2 L}{\partial y^\beta \partial y^\alpha} + y^\beta \left(\rho_\beta^i \frac{\partial^2 L}{\partial x^i \partial y^\alpha} + C_{\alpha\beta}^\gamma \frac{\partial L}{\partial y^\gamma} \right) - \rho_\alpha^i \frac{\partial L}{\partial x^i} = 0.$$

The condition of regularity is equivalent to non-degeneration of the Hessian

$$\varpi_{\alpha\beta} := \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}, \quad |\varpi_{\alpha\beta}| = \det |\varpi_{\alpha\beta}| \neq 0. \quad (10)$$

For regular configurations, we can express the semi-spray vector as

$$\varphi^\varepsilon = \varpi^{\varepsilon\beta} \left(\rho_\beta^i \frac{\partial L}{\partial x^i} - \rho_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\alpha - C_{\beta\alpha}^\gamma \frac{\partial L}{\partial y^\gamma} y^\alpha \right), \quad (11)$$

where $\varpi^{\alpha\beta}$ is inverse to $\varpi_{\alpha\beta}$. If the condition $[\triangle, \Gamma_L]_E = \Gamma_L$ is satisfied, the section Γ_L transforms into a spray which states that the functions φ^β are homogenous of degree 2 on y^β . A curve $c(\tau) = (x^i(\tau), y^\alpha(\tau)) \in E$ for a real parameter τ defines a solution of the Euler–Lagrange equations for L if

$$\frac{dx^i}{d\tau} = \rho_\alpha^i y^\alpha \quad \text{and} \quad \frac{d}{d\tau} \left(\frac{\partial L}{\partial y^\alpha} \right) + y^\beta C_{\alpha\beta}^\gamma \frac{\partial L}{\partial y^\gamma} - \rho_\alpha^i \frac{\partial L}{\partial x^i} = 0. \quad (12)$$

Similarly to the model of Lagrange mechanics on Lie algebroids defined by equations (7) and (12), it is possible to elaborate Hamilton/symplectic geometrizations, see details in [25, 13]. However, in both cases, it is not clear how some versions of Perelman functionals for geometric flows should be performed if we restrict our constructions only to Cartan’s symplectic forms and Lagrangian energy (6) and related equations (7).

3 Lagrangians on Lie Algebroids & N–Connections

In order to elaborate Lagrange–Ricci evolution models on TM and non-holonomic manifolds, we used [9, 10] a geometrization of mechanics in terms of canonical nonlinear and linear connections defined by a regular Lagrangian L . This section is devoted to a brief introduction into the geometry of nonlinear connections on Lie algebroids, see former constructions [16, 18, 19].

3.1 N–connections and prolongations of Lie algebroids

A nonlinear connection, N–connection, structure for a vector bundle P [27] can be defined as a Whitney sum $\mathbf{N} : TP = hTP \oplus vTP$. A

couple $\mathbf{P} := (P, \mathbf{N})$ is called a *nonholonomic vector bundle* (equivalently, vector \mathbf{N} -bundle, with conventional horizontal, h , and vertical, v , splitting/decomposition).³

\mathbf{N} -connections can be similarly introduced on prolongation Lie algebroids via a corresponding h - v -splitting,

$$\mathcal{N} : \mathcal{T}^E \mathbf{P} = h\mathcal{T}^E P \oplus v\mathcal{T}^E P. \quad (13)$$

Such a bundle, and Lie algebroid, morphism $\mathcal{N} : \mathcal{T}^E \mathbf{P} \rightarrow \mathcal{T}^E \mathbf{P}$, with $\mathcal{N}^2 = id$, defines an almost product structure on ${}^P\pi : TP \rightarrow P$ for a smooth map on $TP \setminus \{0\}$, where $\{0\}$ denotes the set of null sections. A \mathbf{N} -connection induces h - and v -projectors for every element $\bar{z} = (p, b, v) \in \mathcal{T}^E P$, when $h(\bar{z}) = {}^h z$ and $v(\bar{z}) = {}^v z$, for $h = \frac{1}{2}(id + \mathcal{N})$ and $v = \frac{1}{2}(id - \mathcal{N})$. These operators define, respectively, the h - and v -subspaces, $h\mathcal{T}^E P = \ker(id - \mathcal{N})$ and $v\mathcal{T}^E P = \ker(id + \mathcal{N})$.

Definition 3.1 A Lie distinguished algebroid (d -algebroid) $\tilde{\mathcal{E}} = (\mathbf{E}, [\cdot, \cdot], \rho)$, is defined for a nonholonomic vector bundle \mathbf{E} endowed with \mathbf{N} -connection structure \mathbf{N} . The prolongations of a Lie algebroid \mathcal{E} over a nonholonomic bundle $\mathbf{P} := (P, \mathbf{N})$ is also a Lie d -algebroid.⁴

Locally, \mathbf{N} -connections are determined respectively by their coefficients $\mathbf{N} = \{N_\alpha^A\}$ and $\mathcal{N} = \{\mathcal{N}_\alpha^A\}$, when

$$\mathbf{N} = N_i^A(x^k, u^B) dx^i \otimes \partial_A \quad \text{and} \quad \mathcal{N} = \mathcal{N}_\alpha^A \mathcal{X}^\alpha \otimes \mathcal{V}_A. \quad (14)$$

Such structures on TP and $\mathcal{T}^E \mathbf{P}$ are compatible if $\mathcal{N}_\alpha^A = N_i^A \rho_\alpha^i$. Using \mathcal{N}_α^A , we can generate sections $\delta_\alpha := \mathcal{X}_\alpha - \mathcal{N}_\alpha^A \mathcal{V}_A$ as a local basis of $h\mathcal{T}^E P$. In general, this allows us to define a \mathbf{N} -adapted frame structure

$$\mathbf{e}_{\bar{\alpha}} := \{\delta_\alpha = \mathcal{X}_\alpha - \mathcal{N}_\alpha^C \mathcal{V}_C, \mathcal{V}_A\}, \quad \text{on } \mathcal{T}^E \mathbf{P}, \quad (15)$$

and its dual

$$\mathbf{e}^{\bar{\beta}} := \{\mathcal{X}^\alpha, \delta^B = \mathcal{V}^B + \mathcal{N}_\gamma^B \mathcal{V}^\gamma\}, \quad (16)$$

³As some particular cases, we can take $P = TM$, for a tangent bundle, or to consider $P = V$, for as a (semi) Riemannian nonholonomic manifold with non-integrable h - v -splitting as we consider in our works on classical and quantum gravity [11, 14, 15]), when "boldface" letters are used for spaces and/or geometric objects defined/adapted to spaces with \mathbf{N} -connection structure.

⁴In our former works, see [18] and references therein, we wrote for a Lie d -algebroid ${}^N\mathcal{E} = (E, \mathbf{N}, [\cdot, \cdot], \rho)$, when left low/up indices are used as abstract labels for some geometric objects and spaces.

where the "overlined" small Greek indices split in the form $\bar{\alpha} = (\alpha, A)$ if an arbitrary vector bundles \mathbf{P} is considered, or $\bar{\alpha} = (\alpha, \alpha)$ if $\mathbf{P} = \mathbf{E}$. The N-adapted bases (15) satisfy certain nonholonomy relations,

$$\mathbf{e}_{\bar{\alpha}} \mathbf{e}_{\bar{\beta}} - \mathbf{e}_{\bar{\beta}} \mathbf{e}_{\bar{\alpha}} = W_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} \mathbf{e}_{\bar{\gamma}},$$

with nontrivial anholonomy coefficients $W_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} = \{C_{\alpha\beta}^{\gamma}, \Omega_{\alpha\beta}^C, \partial_B \mathcal{N}_{\alpha}^C\}$. Such values are determined both by N_{α}^A and Lie algebroid structure constants $C_{\alpha\beta}^{\gamma}$. The corresponding generalized Lie brackets are defined by relations

$$[\delta_{\alpha}, \delta_{\beta}]^{\pi} = C_{\alpha\beta}^{\gamma} \delta_{\gamma} + \Omega_{\alpha\beta}^C \mathcal{V}_C, \quad [\delta_{\alpha}, \mathcal{V}_B]^{\pi} = (\partial_B \mathcal{N}_{\alpha}^C) \mathcal{V}_C, \quad [\mathcal{V}_A, \mathcal{V}_B]^{\pi} = 0.$$

In these formulas, $\Omega_{\alpha\beta}^C = \delta_{\beta} \mathcal{N}_{\alpha}^C - \delta_{\alpha} \mathcal{N}_{\beta}^C + C_{\alpha\beta}^{\gamma} \mathcal{N}_{\gamma}^C$ are the N-adapted coefficients of the Neigenhuis tensor ${}^h N$ of the operator h ,

$${}^h N(\cdot, \cdot) = [h\cdot, h\cdot]^{\pi} - h[h\cdot, \cdot]^{\pi} - h[\cdot, h\cdot]^{\pi} + h^2[h\cdot, h\cdot]^{\pi} = -\frac{1}{2} \Omega_{\alpha\beta}^C \mathcal{X}^{\alpha} \wedge \mathcal{X}^{\beta} \otimes \mathcal{V}_C,$$

which (by definition) is considered to be the curvature of N-connection \mathcal{N}_{α}^A .

It should be noted that for $\mathbf{P} = \mathbf{E}$, the above formulas for Lie d-algebroid $\mathcal{T}^E \mathbf{E}$ mimic on sections of E the geometry of tangent bundles and/or nonholonomic manifolds of even dimension, endowed with N-connection structure (on applications in modern classical and quantum gravity, with various modifications, and nonholonomic Ricci flow theory, see Refs. [14, 9, 11]). If $\mathbf{P} \neq \mathbf{E}$, we model nonholonomic vector bundle and generalized Riemann geometries on sections of $\mathcal{T}^E \mathbf{P}$.

3.2 Linear connections and metrics on $\mathcal{T}^E \mathbf{P}$

The Levi-Civita connection $\varpi \nabla$ (4) on \mathcal{E} is not adapted to a N-connection structure on $\mathcal{T}^E \mathbf{P}$. We have to introduce into consideration another classes of linear connections which would involve the h - v -splitting for $\mathcal{T}^E \mathbf{P}$.

Definition 3.2 *A distinguished connection, d-connection, \mathcal{D} on $\mathcal{T}^E \mathbf{P}$ is a linear connection preserving under parallelism the N-connection (13).*

The N-adapted components $\Gamma_{\bar{\beta}\bar{\gamma}}^{\bar{\alpha}} = (\mathbf{L}_{\beta\gamma}^{\alpha}, \mathbf{L}_{B\gamma}^A; \mathbf{B}_{\beta C}^{\alpha}, \mathbf{B}_{BC}^A)$ of a covariant operator $\mathcal{D}_{\bar{\alpha}} = (\mathbf{e}_{\bar{\alpha}} \rfloor \mathcal{D})$, where \rfloor is the interior product, are computed following equations $\mathcal{D}_{\bar{\alpha}} \mathbf{e}_{\bar{\beta}} = \Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} \mathbf{e}_{\bar{\gamma}}$, or $\Gamma_{\alpha\beta}^{\gamma} = (\mathcal{D}_{\bar{\alpha}} \mathbf{e}_{\bar{\beta}}) \rfloor \mathbf{e}_{\bar{\gamma}}$. The h- and v-covariant derivatives are respectively $h\mathcal{D} = \{\mathcal{D}_{\gamma} = (\mathbf{L}_{\beta\gamma}^{\alpha}, \mathbf{L}_{B\gamma}^A)\}$

and $v\mathcal{D} = \{\mathcal{D}_C = (\mathbf{B}_{\beta C}^\alpha, \mathbf{B}_{BC}^A)\}$, where $\mathbf{L}_{\beta\gamma}^\alpha := (\mathcal{D}_\gamma \delta_\beta)]\mathcal{X}^\alpha$, $\mathbf{L}_{B\gamma}^A := (\mathcal{D}_\gamma \mathcal{V}_B)]\delta^A$, $\mathbf{B}_{\beta C}^\alpha := (\mathcal{D}_C \delta_\beta)]\mathcal{X}^\alpha$, $\mathbf{B}_{BC}^A := (\mathcal{D}_C \mathcal{V}_B)]\delta^A$ are computed for N-adapted bases (15) and (16).

Using rules of absolute differentiation (5) for N-adapted bases $\mathbf{e}_{\bar{\alpha}} := \{\delta_\alpha, \mathcal{V}_A\}$ and $\mathbf{e}^{\bar{\beta}} := \{\mathcal{X}^\alpha, \delta^B\}$ and the d-connection 1-form $\Gamma_{\bar{\alpha}}^{\bar{\gamma}} := \Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} \mathbf{e}^{\bar{\beta}}$, we can compute the torsion and curvature 2-forms on $\mathcal{T}^E \mathbf{P}$:

Let us consider sections $\bar{x}, \bar{y}, \bar{z}$ of $\mathcal{T}^E \mathbf{P}$, were (for instance) $\bar{z} = z^{\bar{\alpha}} \mathbf{e}_{\bar{\alpha}} = z^\alpha \delta_\alpha + z^A \mathcal{V}_A$. The torsion of d-connection \mathcal{D} , $\mathcal{T}(\bar{x}, \bar{y}) = \mathcal{D}_{\bar{x}} \bar{y} - \mathcal{D}_{\bar{y}} \bar{x} + [\bar{x}, \bar{y}]^\pi$ considered as a 2-form is defined as $\mathcal{T}^{\bar{\alpha}} := \mathcal{D} \mathbf{e}^{\bar{\alpha}} = d\mathbf{e}^{\bar{\alpha}} + \Gamma_{\bar{\beta}}^{\bar{\alpha}} \wedge \mathbf{e}^{\bar{\beta}}$. Following a straightforward N-adapted differential form calculus, we prove

Theorem 3.1 *The h-v-coefficients of torsion,*

$\mathcal{T}^{\bar{\alpha}} = \{\mathbf{T}_{\bar{\beta}\bar{\gamma}}^{\bar{\alpha}}\} = \{\mathbf{T}_{\beta\gamma}^\alpha, \mathbf{T}_{\beta A}^\alpha, \mathbf{T}_{\beta\gamma}^A, \mathbf{T}_{B\alpha}^A, \mathbf{T}_{BC}^A\}$, are

$$\begin{aligned} \mathbf{T}_{\beta\gamma}^\alpha &= \mathbf{L}_{\beta\gamma}^\alpha - \mathbf{L}_{\gamma\beta}^\alpha + C_{\beta\gamma}^\alpha, \quad \mathbf{T}_{\beta A}^\alpha = -\mathbf{T}_{A\beta}^\alpha = \mathbf{B}_{\beta A}^\alpha, \quad \mathbf{T}_{\beta\alpha}^A = \Omega_{\beta\alpha}^A, \\ \mathbf{T}_{B\alpha}^A &= \frac{\partial \mathcal{N}_\alpha^A}{\partial u^B} - \mathbf{L}_{B\alpha}^A, \quad \mathbf{T}_{BC}^A = \mathbf{B}_{BC}^A - \mathbf{B}_{CB}^A. \end{aligned} \quad (17)$$

The curvature of \mathcal{D} , $\mathcal{R}(\bar{x}, \bar{y})\bar{z} := (\mathcal{D}_{\bar{x}} \mathcal{D}_{\bar{y}} - \mathcal{D}_{\bar{y}} \mathcal{D}_{\bar{x}} - \mathcal{D}_{[\bar{x}, \bar{y}]^\pi}) \bar{z}$, also can be considered/ computed as a 2-form,

$$\mathcal{R}_{\bar{\beta}}^{\bar{\alpha}} := \mathcal{D} \Gamma_{\bar{\beta}}^{\bar{\alpha}} = d\Gamma_{\bar{\beta}}^{\bar{\alpha}} - \Gamma_{\bar{\beta}}^{\bar{\gamma}} \wedge \Gamma_{\bar{\gamma}}^{\bar{\alpha}} = \mathbf{R}_{\bar{\beta}\bar{\gamma}\bar{\delta}}^{\bar{\alpha}} \mathbf{e}^{\bar{\gamma}} \wedge \mathbf{e}^{\bar{\delta}}, \quad (18)$$

$$\text{where } \mathbf{R}_{\bar{\beta}\bar{\gamma}\bar{\delta}}^{\bar{\alpha}} = \mathbf{e}_{\bar{\delta}} \Gamma_{\bar{\beta}\bar{\gamma}}^{\bar{\alpha}} - \mathbf{e}_{\bar{\gamma}} \Gamma_{\bar{\beta}\bar{\delta}}^{\bar{\alpha}} + \Gamma_{\bar{\beta}\bar{\gamma}}^{\bar{\varphi}} \Gamma_{\bar{\varphi}\bar{\delta}}^{\bar{\alpha}} - \Gamma_{\bar{\beta}\bar{\delta}}^{\bar{\varphi}} \Gamma_{\bar{\varphi}\bar{\gamma}}^{\bar{\alpha}} + \Gamma_{\bar{\beta}\bar{\varphi}}^{\bar{\alpha}} W_{\bar{\gamma}\bar{\delta}}^{\bar{\varphi}}.$$

This results in a proof of

Theorem 3.2 *The curvature of d-connection of \mathcal{D} ,*

$\mathcal{R}_{\bar{\beta}}^{\bar{\alpha}} = \{\mathbf{R}_{\bar{\beta}\bar{\gamma}\bar{\delta}}^{\bar{\alpha}}\} = \{\mathbf{R}_{\varepsilon\beta\gamma}^\alpha, \mathbf{R}_{B\beta\gamma}^A, \mathbf{R}_{\varepsilon\beta A}^\alpha, \mathbf{R}_{B\beta A}^C, \mathbf{R}_{\beta BA}^\alpha, \mathbf{R}_{BEA}^C\}$ is characterized by N-adapted coefficients

$$\begin{aligned} \mathbf{R}_{\varepsilon\beta\gamma}^\alpha &= \delta_\gamma \mathbf{L}_{\varepsilon\beta}^\alpha - \delta_\beta \mathbf{L}_{\varepsilon\gamma}^\alpha + \mathbf{L}_{\varepsilon\beta}^\mu \mathbf{L}_{\mu\gamma}^\alpha - \mathbf{L}_{\varepsilon\gamma}^\mu \mathbf{L}_{\mu\beta}^\alpha + \mathbf{L}_{\varepsilon\varphi}^\alpha C_{\beta\gamma}^\varphi - \mathbf{B}_{\varepsilon A}^\alpha \Omega_{\gamma\beta}^A, \\ \mathbf{R}_{B\beta\gamma}^A &= \delta_\gamma \mathbf{L}_{B\beta}^A - \delta_\beta \mathbf{L}_{B\gamma}^A + \mathbf{L}_{B\beta}^C \mathbf{L}_{C\gamma}^A - \mathbf{L}_{B\gamma}^C \mathbf{L}_{C\beta}^A + \mathbf{L}_{B\varphi}^A C_{\beta\gamma}^\varphi - \mathbf{B}_{BC}^A \Omega_{\gamma\beta}^C, \\ \mathbf{R}_{\varepsilon\beta A}^\alpha &= \mathcal{V}_A \mathbf{L}_{\varepsilon\beta}^\alpha - \mathcal{D}_\beta \mathbf{B}_{\varepsilon A}^\alpha + \mathbf{B}_{\varepsilon B}^\alpha \mathbf{T}_{\beta A}^B, \\ \mathbf{R}_{B\gamma A}^C &= \mathcal{V}_A \mathbf{L}_{B\gamma}^C - \mathcal{D}_\gamma \mathbf{B}_{BA}^C + \mathbf{B}_{BD}^C \mathbf{T}_{\gamma A}^D, \\ \mathbf{R}_{\beta BA}^\alpha &= \mathcal{V}_A \mathbf{B}_{\beta B}^\alpha - \mathcal{V}_B \mathbf{B}_{\beta C}^\alpha + \mathbf{B}_{\beta B}^\tau \mathbf{B}_{\tau C}^\alpha - \mathbf{B}_{\beta C}^\tau \mathbf{B}_{\tau B}^\alpha, \\ \mathbf{R}_{ECB}^A &= \mathcal{V}_E \mathbf{B}_{BC}^A - \mathcal{V}_C \mathbf{B}_{BE}^A + \mathbf{B}_{BC}^F \mathbf{B}_{FE}^A - \mathbf{B}_{BE}^F \mathbf{B}_{FC}^A. \end{aligned} \quad (19)$$

We note that in above first two formulas the terms $\mathbf{L}_{\varepsilon\varphi}^\alpha C_{\beta\gamma}^\varphi$ and $\mathbf{L}_{B\varphi}^A C_{\beta\gamma}^\varphi$, respectively, transform in zero for a trivial Lie algebroid commutator structure when $C_{\beta\gamma}^\varphi = 0$. In such a case, the geometry of $\mathcal{T}^E \mathbf{P}$ endowed with

N-connection structure \mathcal{N}_γ^C mimics a similar one for the associated vector bundle \mathbf{P} with a nontrivial N_i^α . Using prolongations of Lie algebroids on fibration maps, we model tangent bundle geometries but not in a complete equivalent form because there are differences in chosen nonholonomic structures and torsions and curvatures of d-connections.

Corollary 3.1 *The Ricci tensor of \mathcal{D} , $\text{Ric} = \{\mathbf{R}_{\bar{\alpha}\bar{\beta}} := \mathbf{R}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}^{\bar{\gamma}}\}$, is characterized by N-adapted coefficients*

$$\mathbf{R}_{\bar{\alpha}\bar{\beta}} = \{\mathbf{R}_{\alpha\beta} := \mathbf{R}_{\alpha\beta\chi}^\chi, \mathbf{R}_{\alpha A} := -\mathbf{R}_{\alpha\gamma A}^\gamma, \mathbf{R}_{A\alpha} := \mathbf{R}_{A\alpha B}^B, \mathbf{R}_{AB} := \mathbf{R}_{ABC}^C\}. \quad (20)$$

Proof. The formulas for h - v -components (20) are respective contractions of the coefficients (19). \square

Definition 3.3 *A metric structure on $\mathcal{T}^E\mathbf{P}$ is defined by a nondegenerate symmetric second rank tensor $\bar{\mathbf{g}} = \{\mathbf{g}_{\bar{\alpha}\bar{\beta}}\}$. Such a tensor is called a distinguished metric, i.e. a d -metric, if its coefficients are defined with respect to tensor products of N-adapted frames (16),*

$$\bar{\mathbf{g}} = \mathbf{g}_{\bar{\alpha}\bar{\beta}} \mathbf{e}^{\bar{\beta}} \otimes \mathbf{e}^{\bar{\beta}} = \mathbf{g}_{\alpha\beta} \mathcal{X}^\alpha \otimes \mathcal{X}^\beta + \mathbf{g}_{AB} \delta^A \otimes \delta^B. \quad (21)$$

We can define the inverse d -metric $\bar{\mathbf{g}}^{\bar{\alpha}\bar{\beta}}$ and inverse N-adapted h -metric, $\mathbf{g}^{\alpha\beta}$, and v -metric, \mathbf{g}^{AB} , by inverting respectively the matrix $\mathbf{g}_{\bar{\alpha}\bar{\beta}}$ and its bloc components, $\mathbf{g}_{\alpha\beta}$ and \mathbf{g}_{AB} .

The scalar curvature ${}^s\mathbf{R}$ of \mathcal{D} is by definition

$${}^s\mathbf{R} := \bar{\mathbf{g}}^{\bar{\alpha}\bar{\beta}} \mathbf{R}_{\bar{\alpha}\bar{\beta}} = \mathbf{g}^{\alpha\beta} \mathbf{R}_{\alpha\beta} + \mathbf{g}^{AB} \mathbf{R}_{AB}. \quad (22)$$

Using (20) and (22), we can compute the Einstein tensor $\mathbf{E}_{\bar{\alpha}\bar{\beta}}$ of \mathcal{D} ,

$$\mathbf{E}_{\bar{\alpha}\bar{\beta}} \doteq \mathbf{R}_{\bar{\alpha}\bar{\beta}} - \frac{1}{2} \bar{\mathbf{g}}_{\bar{\alpha}\bar{\beta}} {}^s\mathbf{R}. \quad (23)$$

Such a tensor can be used for modeling effective gravity theories on sections of $\mathcal{T}^E\mathbf{P}$ with nonholonomic frame structure [14, 15, 18, 19].

3.3 Metric compatible geometries on Lie d -algebroids

Additionally to torsion (17) and curvature (19), d -connections are characterized by nonmetricity field $\mathcal{Q}(\bar{y}) := \mathcal{D}_{\bar{y}}\bar{\mathbf{g}}$, when $\mathbf{Q}_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} = \mathcal{D}^{\bar{\gamma}}\bar{\mathbf{g}}_{\bar{\alpha}\bar{\beta}}$.

Proposition 3.1 *The condition of metric compatibility, $\mathcal{Q} = \mathcal{D}\bar{\mathbf{g}} = 0$, splits into respective conditions for h - v -components, $\mathcal{D}_\gamma \mathbf{g}_{\alpha\beta} = 0, \mathcal{D}_A \mathbf{g}_{\alpha\beta} = 0, \mathcal{D}_\gamma \mathbf{g}_{AB} = 0, \mathcal{D}_C \mathbf{g}_{AB} = 0$.*

Proof. It follows from a straightforward computation when the coefficients of d-metric $\mathbf{g}_{\bar{\alpha}\bar{\beta}}$ (21) are introduced into $\mathcal{D}_{\bar{y}}\bar{\mathbf{g}} = 0$, for $\bar{y} = y^{\bar{\alpha}}\mathbf{e}_{\bar{\alpha}} = y^\alpha\delta_\alpha + y^A\mathcal{V}_A$. \square

In this paper, we shall work with two "preferred" linear connections completely defined by a d-metric structure $\bar{\mathbf{g}}$ on $\mathcal{T}^E\mathbf{P}$:

Theorem 3.3 *There is a canonical d-connection $\hat{\mathcal{D}}$ for which $\hat{\mathcal{D}}\bar{\mathbf{g}} = 0$ and h - and v -torsions (17) are prescribed, respectively, to be with coefficients $\hat{T}^\alpha_{\beta\gamma} = C^\alpha_{\beta\gamma}$ and $\hat{T}^A_{BC} = 0$ computed with respect to N -adapted frames.*

Proof. We can check by straightforward computations that the conditions of this theorem are satisfied if and only if $\hat{\mathcal{D}}$ is taken with N -adapted coefficients $\hat{\Gamma}^\alpha_{\bar{\beta}\bar{\gamma}} = (\hat{\mathbf{L}}^\alpha_{\beta\gamma}, \hat{\mathbf{L}}^A_{B\gamma}; \hat{\mathbf{B}}^\alpha_{\beta C}, \hat{\mathbf{B}}^A_{BC})$ for

$$\begin{aligned}\hat{L}^\alpha_{\beta\gamma} &= \frac{1}{2}\mathbf{g}^{\alpha\tau}(\delta_\gamma\mathbf{g}_{\beta\tau} + \delta_\beta\mathbf{g}_{\gamma\tau} - \delta_\tau\mathbf{g}_{\beta\gamma}) + \frac{1}{2}\mathbf{g}^{\alpha\tau}(\mathbf{g}_{\beta\varepsilon}C^\varepsilon_{\tau\gamma} + \mathbf{g}_{\gamma\varepsilon}C^\varepsilon_{\tau\beta} - \mathbf{g}_{\tau\varepsilon}C^\varepsilon_{\beta\gamma}), \\ \hat{L}^A_{B\gamma} &= \mathcal{V}_B(\mathcal{N}_\gamma^A) + \frac{1}{2}\mathbf{g}^{AC}(\delta_\gamma\mathbf{g}_{BC} - \mathbf{g}_{DC}\mathcal{V}_B(\mathcal{N}_\gamma^D) - \mathbf{g}_{DB}\mathcal{V}_C(\mathcal{N}_\gamma^D)), \\ \hat{B}^\alpha_{\beta C} &= \frac{1}{2}\mathbf{g}^{\alpha\tau}\mathcal{V}_C\mathbf{g}_{\beta\tau}, \quad \hat{B}^A_{BC} = \frac{1}{2}\mathbf{g}^{AD}(\mathcal{V}_C\mathbf{g}_{BD} + \mathcal{V}_B\mathbf{g}_{CD} - \mathcal{V}_D\mathbf{g}_{BC}). \quad \square\end{aligned}\tag{24}$$

The nontrivial values of torsion of $\hat{\mathcal{D}}$, i.e. N -adapted coefficients $\hat{T}^\alpha_{\beta\gamma}$, $\hat{T}^\alpha_{\beta A}$, $\hat{T}^A_{\beta\alpha}$ and $\hat{T}^A_{B\alpha}$, are computed by introducing the canonical d-connection coefficients (24) into formulas (17).

Theorem 3.4 (–Definition) *There is a metric compatible Levi-Civita connection $\bar{\nabla}$ which is completely defined by a d-metric structure $\bar{\mathbf{g}}$ on $\mathcal{T}^E\mathbf{P}$ following the condition of zero torsion, ${}^\nabla\mathcal{T}^\alpha = \{K^\alpha_{\bar{\beta}\bar{\gamma}}\} = 0$.*

Proof. Such a connection $\bar{\nabla} = K^\alpha_{\bar{\beta}\bar{\gamma}} = (\bar{L}^\alpha_{\beta\gamma}, \bar{L}^A_{B\gamma}; \bar{B}^\alpha_{\beta C}, \bar{B}^A_{BC})$ can be defined with respect to N -adapted frames for the same d-metric structure $\bar{\mathbf{g}}$ which is used for constructing $\hat{\mathcal{D}}$ (24), but with additional constraints that all torsion coefficients (17) are zero. We can verify via straightforward computations with respect to (15) and (16) that the condition of theorem is satisfied by a distortion relation

$$K^\alpha_{\bar{\beta}\bar{\gamma}} = \hat{\Gamma}^\alpha_{\bar{\beta}\bar{\gamma}} + \hat{\mathbf{Z}}^\alpha_{\bar{\beta}\bar{\gamma}}, \tag{25}$$

where the distortion tensor $\hat{\mathcal{Z}} = \{\hat{\mathbf{Z}}^\gamma_{\alpha\beta}\}$ is given by N-adapted coefficients

$$\begin{aligned}\hat{\mathbf{Z}}^A_{\beta\gamma} &= -\hat{\mathbf{B}}^\alpha_{\beta B} \mathbf{g}_{\alpha\gamma} \mathbf{g}^{AB} - \frac{1}{2} \Omega^A_{\beta\gamma} \hat{\mathbf{Z}}^\alpha_{B\gamma} = \frac{1}{2} \Omega^C_{\alpha\gamma} \mathbf{g}_{CB} \mathbf{g}^{\beta\alpha} - \Xi^{\alpha\tau}_{\beta\gamma} \hat{\mathbf{B}}^\beta_{\tau B}, \\ \hat{\mathbf{Z}}^A_{B\gamma} &= +\Xi^{AB}_{CD} \hat{\mathbf{T}}^C_{\gamma B}, \quad \hat{\mathbf{Z}}^i_{\gamma B} = \frac{1}{2} \Omega^A_{\beta\gamma} \mathbf{g}_{CB} \mathbf{g}^{\beta\alpha} + \Xi^{\alpha\tau}_{\beta\gamma} \hat{\mathbf{B}}^\beta_{\tau B}, \quad \hat{\mathbf{Z}}^\alpha_{\beta\gamma} = 0, \\ \hat{\mathbf{Z}}^A_{\beta B} &= -\Xi^{AD}_{CB} \hat{\mathbf{T}}^C_{\beta D}, \quad \hat{\mathbf{Z}}^A_{BC} = 0, \quad \hat{\mathbf{Z}}^\alpha_{AB} = -\frac{\mathbf{g}^{\alpha\beta}}{2} \left[\hat{\mathbf{T}}^C_{\beta A} \mathbf{g}_{CB} + \hat{\mathbf{T}}^C_{\beta B} \mathbf{g}_{CA} \right],\end{aligned}\tag{26}$$

for $\Xi^{\alpha\tau}_{\beta\gamma} = \frac{1}{2}(\delta^\alpha_\beta \delta^\tau_\gamma - \mathbf{g}_{\beta\gamma} \mathbf{g}^{\alpha\tau})$ and $\pm \Xi^{AB}_{CD} = \frac{1}{2}(\delta^A_C \delta^B_D \pm \mathbf{g}_{CD} \mathbf{g}^{AB})$.

The distortion coefficients (26) are such linear algebraic combinations of coefficients of torsion of $\hat{\mathcal{D}}$ that the condition $\hat{\mathbf{T}}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = 0$ is equivalent to $\hat{\mathbf{Z}}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = 0$, and inversely. So, we can find a h - v -decomposition when $\bar{\Gamma}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = \hat{\Gamma}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}$ even, in general, $\bar{\nabla} \neq \hat{\mathcal{D}}$; such connections are subjected to different rules of frame/coordinate transforms on $\mathcal{T}^E \mathbf{P}$. \square

We emphasize that $\bar{\nabla}$ is not a d-connection and does not preserve under parallelism the N-connection structure. Nevertheless, all geometric data for $(\bar{\mathbf{g}}, \bar{\nabla})$ can be transformed equivalently into similar ones for $(\bar{\mathbf{g}}, \hat{\mathcal{D}}, \mathcal{N})$ when $\bar{\mathbf{g}}$ and \mathcal{N} define a unique N-adapted splitting $\bar{\nabla} = \hat{\mathcal{D}} + \hat{\mathcal{Z}}$.⁵

Corollary 3.2 *Any metric $\bar{\mathbf{g}}$ on $\mathcal{T}^E \mathbf{P}$ can be represented equivalently as a d-metric $\mathbf{g}_{\bar{\alpha}\bar{\beta}}$ (21) or, with respect to a local dual base $dz^{\bar{\beta}} := \{\mathcal{X}^\alpha, \mathcal{V}^B\}$, in generic off-diagonal form, $\mathbf{g} = g_{\bar{\alpha}\bar{\beta}} dz^{\bar{\alpha}} \otimes dz^{\bar{\beta}}$, with "non-boldface" coefficients*

$$g_{\bar{\alpha}\bar{\beta}} = \begin{bmatrix} \mathbf{g}_{\alpha\beta} + \mathcal{N}^A_\alpha \mathcal{N}^B_\beta & \mathbf{g}_{AB} & \mathcal{N}^A_\beta \mathbf{g}_{AC} \\ \mathcal{N}^E_\alpha \mathbf{g}_{ED} & & \mathbf{g}_{DC} \end{bmatrix}.\tag{27}$$

Proof. A frame transform $\mathbf{e}^{\bar{\beta}} \rightarrow dz^{\bar{\beta}}$ is dual to matrix transform $\partial_{\bar{\alpha}'} \rightarrow \mathbf{e}_{\bar{\alpha}}^{\bar{\alpha}'} \partial_{\bar{\alpha}'}$, with

$$\mathbf{e}_{\bar{\alpha}}^{\bar{\alpha}'} = \begin{bmatrix} \mathbf{e}_\alpha^{\alpha'} & \mathcal{N}^B_\alpha \mathbf{e}_B^{A'} \\ 0 & \mathbf{e}_A^{A'} \end{bmatrix}.\tag{28}$$

Additionally, one should be considered some quadratic relations between coefficients $g_{\bar{\alpha}\bar{\beta}} = e_{\bar{\alpha}}^{\bar{\alpha}'} \frac{\bar{\beta}'}{\beta} \eta_{\bar{\alpha}'\bar{\beta}'}$, for $\eta_{\bar{\alpha}'\bar{\beta}'} = \text{diag}[\pm 1, \dots, \pm 1]$ fixing a local signature for metric on $\mathcal{T}^E \mathbf{P}$. A metric (27) is called generic off-diagonal because it can not be diagonalized by coordinate transforms. \square

⁵By geometric data, we consider any set of geometric tensors, forms, connections etc and relevant field/evolution/ constraint equations which can be used in a model of geometry and/or physical theory.

Remark 3.1 Introducing $K_{\beta\gamma}^{\bar{\alpha}} = \hat{\Gamma}_{\beta\gamma}^{\bar{\alpha}}$ (24) into formulas (19), (20) and (22), we compute respectively the coefficients of curvature, $\hat{\mathbf{R}}_{\beta\gamma\delta}^{\bar{\alpha}}$, Ricci tensor, $\hat{\mathbf{R}}_{\alpha\beta}$, and scalar curvature, ${}^s\hat{\mathbf{R}}$. The distortions $K = \hat{\Gamma} + \hat{\mathbf{Z}}$ (25) allows us to compute the distorting tensors ($\hat{\mathbf{Z}}_{\beta\gamma\delta}^{\bar{\alpha}}$, $\hat{\mathbf{Z}}_{\alpha\beta}$ and ${}^s\hat{\mathbf{Z}}$) resulting in similar values for the (pseudo) Riemannian geometry on $\mathcal{T}^E\mathbf{P}$ determined by $(\bar{\mathbf{g}}, K)$, i.e. to define $R_{\beta\gamma\delta}^{\bar{\alpha}}$, $R_{\beta\gamma}$ and sR .

We do not present all technical details and component formulas for geometrical objects outlined in above Remark. As an example, we provide the distortion relations for the Ricci tensor,

$$R_{\alpha\beta} = \hat{\mathbf{R}}_{\alpha\beta} + \hat{\mathbf{Z}}_{\alpha\beta}, \quad (29)$$

$$\begin{aligned} R_{\beta\gamma}^{\bar{\alpha}} &= R_{\beta\gamma\alpha}^{\bar{\alpha}} = \mathbf{e}_{\delta}^{\bar{\alpha}} K_{\beta\gamma}^{\bar{\alpha}} - \mathbf{e}_{\gamma}^{\bar{\alpha}} K_{\beta\delta}^{\bar{\alpha}} + K_{\beta\gamma}^{\bar{\varphi}} K_{\bar{\varphi}\delta}^{\bar{\alpha}} - K_{\beta\delta}^{\bar{\varphi}} K_{\bar{\varphi}\gamma}^{\bar{\alpha}} + K_{\beta\bar{\varphi}}^{\bar{\alpha}} W_{\bar{\varphi}\delta}^{\bar{\alpha}}, \\ \hat{\mathbf{Z}}_{\beta\gamma}^{\bar{\alpha}} &= \hat{\mathbf{Z}}_{\beta\gamma\alpha}^{\bar{\alpha}} = \mathbf{e}_{\alpha}^{\bar{\alpha}} \hat{\mathbf{Z}}_{\beta\gamma}^{\bar{\alpha}} - \mathbf{e}_{\gamma}^{\bar{\alpha}} \hat{\mathbf{Z}}_{\beta\alpha}^{\bar{\alpha}} + \hat{\mathbf{Z}}_{\beta\gamma}^{\bar{\varphi}} \hat{\mathbf{Z}}_{\bar{\varphi}\alpha}^{\bar{\alpha}} - \hat{\mathbf{Z}}_{\beta\alpha}^{\bar{\varphi}} \hat{\mathbf{Z}}_{\bar{\varphi}\gamma}^{\bar{\alpha}} + \\ &\quad \hat{\Gamma}_{\beta\gamma}^{\bar{\varphi}} \hat{\mathbf{Z}}_{\bar{\varphi}\alpha}^{\bar{\alpha}} - \hat{\Gamma}_{\beta\alpha}^{\bar{\varphi}} \hat{\mathbf{Z}}_{\bar{\varphi}\gamma}^{\bar{\alpha}} + \hat{\mathbf{Z}}_{\beta\gamma}^{\bar{\varphi}} \hat{\Gamma}_{\bar{\varphi}\alpha}^{\bar{\alpha}} - \hat{\mathbf{Z}}_{\beta\alpha}^{\bar{\varphi}} \hat{\Gamma}_{\bar{\varphi}\gamma}^{\bar{\alpha}} + \hat{\mathbf{Z}}_{\beta\bar{\varphi}}^{\bar{\alpha}} W_{\bar{\varphi}\alpha}^{\bar{\alpha}}. \end{aligned}$$

Such values are defined with respect to N-adapted bases (15) and (16). Using fame transforms (28) and their dual ones computed as inverse matrices $(\mathbf{e}_{\bar{\alpha}}^{\bar{\alpha}'})^{-1}$, we can re-define the coefficients with respect to coordinate bases. Coordinate formulas are important in the theory of Ricci flows (allowing simplified proofs of a number of important results on geometric evolution) and for constructing, in explicit form, exact solutions in geometric mechanics and analogous gravity.

Finally, we note that all values on prolongation Lie algebroids are uniquely determined by a d-metric $\mathbf{g}_{\alpha\beta}$ (21) (equivalently, by a generic off-diagonal $g_{\alpha\beta}$ (27)) for a prescribed \mathcal{N}_{α}^B (13). Elaborating a physical dynamical/evolution model for $(\bar{\mathbf{g}}, \bar{\nabla})$, the same theory can be described in terms of data $(\bar{\mathbf{g}}, \hat{\mathcal{D}}, \mathcal{N})$. This property allows us to simplify, for instance, the proofs of main results for Ricci flows on Lie algebroids using similar results for (pseudo) Riemannian metrics and then nonholonomically transforming the constructions into evolution of N-adapted values.

3.4 An extension of Kern–Matsumoto approach for algebroid mechanics & gravity

Let us consider an alternative (second) approach to geometrization of regular Lagrange mechanics [16, 17] on Lie algebroids. All constructions

are described in terms of generalized metrics, adapted frames and N- and d-connections. This is different from Cartan variables (6) and equations (8) considered in section 2.⁶

The goal of this section is to show that it is possible a setting when canonical N- and d-connections and d-metric on $\mathcal{T}^E \mathbf{E}$, for $\mathbf{P} = \mathbf{E}$, considered in previous section, are derived from a regular Lagrangian as a solution of Euler-Lagrange equations (12).

Lemma 3.1 *There is a N-connection ${}^q\mathcal{N} := -\mathcal{L}_q S$ defined by a semi-spray $q = y^\alpha \mathcal{X}_\alpha + q^\alpha \mathcal{V}_\alpha$ and Lie derivative \mathcal{L}_q acting on any $X \in \text{Sec}(TE)$ following formula ${}^q\mathcal{N}(X) = -[q, SX]^\pi + S[q, X]^\pi$.*

Proof. Consider the operators S and Δ from (9) defining the semi-spray via formula $Sq = \Delta$. For local coordinates $(x^i, u^A) \rightarrow (x^i, y^\alpha)$ and $X = \mathcal{X}_\alpha$, we compute ${}^q\mathcal{N}(\mathcal{X}_\alpha) = -[q, S(\mathcal{X}_\alpha)]^\pi + S[q, \mathcal{X}_\alpha]^\pi = \mathcal{X}_\alpha + (\partial_\alpha q^\beta + y^\gamma C_{\gamma\alpha}^\beta) \mathcal{V}_\beta$. Using ${}^q\mathcal{N}(\mathcal{V}_\alpha) = -\mathcal{V}_\alpha$ and ${}^q\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - 2 {}^q\mathcal{N}_\alpha^\gamma(x, y) \mathcal{V}_\gamma$, we define the N-connection coefficients $\mathcal{N}_\alpha^\gamma = -\frac{1}{2} (\partial_\alpha q^\gamma + y^\beta C_{\beta\alpha}^\gamma)$, see formulas (14). \square

To generate N-connections, we can use sections $\Gamma_L = y^\alpha \mathcal{X}_\alpha + \varphi^\alpha \mathcal{V}_\alpha$, with $q^\varepsilon = \varphi^\varepsilon(x^i, y^\beta)$ (11).

Theorem 3.5 *Any regular Lagrangian $L \in C^\infty(E)$ defines a canonical N-connection on prolongation Lie algebroid $\mathcal{T}^E \mathbf{E}$,*

$$\tilde{\mathcal{N}} = {}^\varphi\mathcal{N} = \{\tilde{\mathcal{N}}_\alpha^\gamma = -\frac{1}{2} (\partial_\alpha \varphi^\gamma + y^\beta C_{\beta\alpha}^\gamma)\}, \quad (30)$$

determined by semi-spray configurations encoding the solutions of Euler-Lagrange equations (12).

Proof. It is a straightforward consequence of above Lemma and (11). \square

The geometric data and dynamics of symplectic equations (8) for Cartan variables (6) can be encoded equivalently into a metric compatible geometry on prolongation of Lie algebroid.

Corollary 3.3 *A model of Lie algebroid geometry $(L : \tilde{\mathcal{N}}, \tilde{\mathbf{g}}, \hat{\mathcal{D}})$ on $\mathcal{T}^E \mathbf{E}$, for $\pi : E \rightarrow M$, with prescribed algebroid structure functions $\rho_\alpha^i(x^k)$ and $C_{\beta\alpha}^\gamma(x^k)$, is canonically determined by a regular Lagrangian $L \in C^\infty(E)$.*

⁶To develop such ideas for mathematical relativity and geometric mechanics and non-commutative and Lie algebroid modifications/ generalizations was proposed in some our proposals for Projects and Marie Curies Fellowships in 2003–2004, see also papers [18, 19] and references therein.

Proof. It follows from such key steps in definition of fundamental geometric objects. Using $L(x, y)$, we construct the canonical N-connection $\tilde{\mathcal{N}} = \{\tilde{\mathcal{N}}_\alpha^\gamma\}$ (30) and induced N-adapted frames (15) and (16), respectively,

$$\tilde{\mathbf{e}}_\alpha := \{\delta_\alpha = \mathcal{X}_\alpha - \tilde{\mathcal{N}}_\alpha^\gamma \mathcal{V}_\gamma, \mathcal{V}_\beta\} \quad \text{and} \quad \tilde{\mathbf{e}}^{\bar{\beta}} := \{\mathcal{X}^\alpha, \delta^\beta = \mathcal{V}^\beta + \tilde{\mathcal{N}}_\gamma^\beta \mathcal{V}^\gamma\}. \quad (31)$$

At the next step, we construct a total metric of type (21), $\bar{\mathbf{g}} \rightarrow \tilde{\mathbf{g}}$, as a Sasaki lift of Hessian $\varpi_{\alpha\beta}$ (10), where

$$\tilde{\mathbf{g}} := \tilde{\mathbf{g}}_{\bar{\alpha}\bar{\beta}} \mathbf{e}^{\bar{\beta}} \otimes \mathbf{e}^{\bar{\alpha}} = \varpi_{\alpha\beta} \mathcal{X}^\alpha \otimes \mathcal{X}^\beta + \varpi_{\alpha\beta} \delta^\alpha \otimes \delta^\beta. \quad (32)$$

Introducing the coefficients of d-metric (32) into formulas (24), we compute the coefficients of canonical $\hat{\mathcal{D}}$, induced by L . \square

In general, we can use arbitrary frames of reference on $\mathcal{T}^E \mathbf{E}$, when $\mathbf{e}_{\bar{\gamma}'} = e^{\bar{\gamma}}_{\bar{\gamma}'} \tilde{\mathbf{e}}_{\bar{\gamma}}$ for any $\tilde{\mathbf{e}}_{\bar{\gamma}}$ (31). Any N-connection and/or metric structure $(\mathcal{N}, \bar{\mathbf{g}})$ can be related to some canonical data $(\tilde{\mathcal{N}}, \tilde{\mathbf{g}})$ determined by a regular Lagrangian $L \in C^\infty(E)$, when $\bar{\mathbf{g}}_{\bar{\alpha}'\bar{\beta}'} = e^{\bar{\alpha}}_{\bar{\alpha}'} e^{\bar{\beta}}_{\bar{\beta}'} \tilde{\mathbf{g}}_{\bar{\alpha}\bar{\beta}}$. It is necessary to solve an algebraic quadratic system of equations in order to define $e^{\bar{\alpha}}_{\bar{\alpha}'}$ from some prescribed data $\bar{\mathbf{g}}_{\bar{\alpha}'\bar{\beta}'}$ and $\tilde{\mathbf{g}}_{\bar{\alpha}\bar{\beta}}$.

Conclusion 3.1 *Via corresponding frame transforms and re-adapting non-holonomic distributions on $\mathcal{T}^E \mathbf{E}$, we can model equivalently:*

- *any Lagrange mechanics determined by regular $L \in C^\infty(E)$ as a Kern–Matsumoto model $(L : \tilde{\mathcal{N}}, \tilde{\mathbf{g}}, \hat{\mathcal{D}})$ and corresponding Ricci tensor, $\hat{\mathcal{R}}ic = \{\hat{\mathbf{R}}_{\bar{\alpha}\bar{\beta}}\}$ (20), and scalar curvature ${}^s\hat{\mathbf{R}}$ (22);*
- *inversely, any off-diagonal metric $g_{\bar{\alpha}\bar{\beta}}$ (27) can be transformed via N-adapted frame transforms (28) into a d-metric $\mathbf{g}_{\bar{\alpha}\bar{\beta}}$ (21) (for a prescribed L , parametrized in a form $\tilde{\mathbf{g}}$ (32)); we can model analogous gravity theories on algebroids as effective Lagrange models.*

Following different approaches, algebroid models for analogous gravity and matter field interactions are studied in Refs. [14, 15, 18, 19, 26]. One of the most important problems for such theories is to provide a physical motivation for the type of linear connection which should be chosen for constructing analogs of Einstein equations of Lie algebroids and how solutions of gravitational field equations are related to the Euler–Lagrange equations in an effective geometric mechanics. For the purposes of this paper, it is

important to consider the case of Einstein N-anholonomic spaces on Lie algebroid prolongations defined by solutions of equations

$$\hat{\mathbf{R}}_{\alpha\bar{\beta}} = \lambda \mathbf{g}_{\alpha\bar{\beta}} \quad (33)$$

where $\lambda = \text{const.}$ Such configurations are stationary ones in the theory of Ricci flows and define the so-called Ricci solitons. In the case of Lie algebroids determined by $\pi : E \rightarrow M$, with local coordinates (x^i, y^α) , the solutions for effective metrics induced on sections are with Killing symmetries on $\partial/\partial y^\alpha$, when Lie derivatives of $\mathbf{g}_{\alpha\bar{\beta}}$ on such y -directions are zero, because all algebroid geometric structures are defined as sections over M (all structure functions and coefficients of fundamental geometric objects depending on local coordinates x^i).

Claim 3.1 *Stationary (with respect to Ricci flow evolution of geometric structure on Lie algebroids, see next section) effective Lagrange and/or analogous gravitational models on a prolongation Lie algebroid $\mathcal{T}^E \mathbf{E}$ for $\pi : E \rightarrow M$, $\dim E = n + m$ and $\dim M = n \geq 2$, are defined as Ricci soliton configurations (33) with m Killing symmetries.*

For certain classes of smooth functions, such a Claim can be proven using theorems on decoupling and integration of the Einstein–Yang–Mills–Higgs equations, [14, 31]. Nevertheless, this Claim can not be proven for all possible types of Lie algebroid configurations. The gravitational and matter field equations on different curved spaces, including constructions with Lie algebroids, are very sophisticated nonlinear systems of partial differential equations. In general, such systems may have various stochastic, fractional, chaos etc properties. This gives us a reason to argue that following our experience a chosen class of Cauchy type and/or stochastic etc flows can be modelled by a corresponding effective Lagrange dynamics/ evolution of Lie algebroid configurations. We can not prove that all physically important cases can be described via such models and it is not possible to state some uniqueness criteria, completeness of solutions etc.

4 Lagrange–Ricci Evolution and Lie Algebroids

Following Kern–Matsumoto geometrization of regular Lagrange mechanics and analogous gravity models on Lie algebroids, we can consider the problem of geometric flow evolution of such system as an explicit example of a theory of Ricci flows on nonholonomic manifolds as we stated in Refs.

[9, 10, 11]. The goal of this section is to prove that Lagrange–Ricci flows on $\mathcal{T}^E \mathbf{E}$ can be encoded into a model of gradient nonholonomic flows.

We can formulate an evolution model for a family of geometric data $(\bar{\mathbf{g}}(\tau), \bar{\nabla}(\tau))$ on $\mathcal{T}^E \mathbf{E}$ induced by a family of regular $L(\tau) \in C^\infty(E)$ with a flow parameter $\tau \in [-\epsilon, \epsilon] \subset \mathbb{R}$, when $\epsilon > 0$ is taken sufficiently small. Let us introduce on the space of $Sec(E)$, for $\pi : E \rightarrow M$, $\dim E = n + m$ and $\dim M = n \geq 2$, the functionals

$$\begin{aligned} {}_i\mathcal{F}(\bar{\mathbf{g}}, \bar{\nabla}, f, \tau) &= \int_{\bar{\mathcal{V}}} \left({}_iR + |\bar{\nabla}f|^2 \right) e^{-f} dV, \\ {}_i\mathcal{W}(\bar{\mathbf{g}}, \bar{\nabla}, f, \tau) &= \int_{\bar{\mathcal{V}}} \left[\tau \left({}_iR + |\bar{\nabla}f|^2 \right) + f - 2m \right] \mu dV, \end{aligned} \quad (34)$$

where the volume form dV and scalar curvature ${}_iR$ are determined by an off-diagonal metric $g_{\bar{\alpha}\bar{\beta}}$ (27). The integration is taken over $\bar{\mathcal{V}} \subset \mathcal{T}^E \mathbf{E}$, $\dim \mathcal{V} = 2m$, corresponding to sections over a $U \subset M$. We can fix $\int_{\bar{\mathcal{V}}} dV = 1$, with $\mu = (4\pi\tau)^{-m} e^{-f}$, considering necessary classes of frame transforms and a parameter $\tau > 0$. The Ricci flow evolution derived from (34) in variables $(\bar{\mathbf{g}}, \bar{\nabla})$ is a standard theory for Riemann metrics [1, 2, 3, 4, 5] but restricted to the conditions that such metrics are induced by regular Lagrangians. The evolution in such variables is not adapted to a N-connection structure (13). It is possible to elaborate N-adapted scenarios if above Perelman's functionals are re-defined in terms of geometric data $(\tilde{\mathbf{g}}, \hat{\mathcal{D}})$ and the derived flow equations are considered in N-adapted variables. Both approaches are equivalent if the distortion relations $\bar{\nabla} = \hat{\mathcal{D}} + \hat{\mathbf{Z}}$ (25) are considered for the same family of metrics, $\bar{\mathbf{g}}(\tau) = \tilde{\mathbf{g}}(\tau)$ computed for the same set $L(\tau)$.

The theory of Lagrange–Ricci flows on $\mathcal{T}^E \mathbf{E}$ is formulated as a model of evolving nonholonomic dynamical systems on the space of equivalent geometric data $(L : \bar{\mathbf{g}}, \bar{\nabla})$ and/or $(L : \tilde{\mathbf{g}}, \hat{\mathcal{D}})$ when the functionals ${}_i\mathcal{F}$ and ${}_i\mathcal{W}$ are postulated to be of Lyapunov type. Ricci flat configurations (the Ricci tensor can be computed for one of the connections $\bar{\nabla}$ or $\hat{\mathcal{D}}$) are defined as "fixed" on τ points of the corresponding dynamical systems.

We use $\check{\tau} = {}^h\tau = {}^v\tau$ for a couple of possible h - and v -flows parameters, $\check{\tau} = ({}^h\tau, {}^v\tau)$, and introduce a new function \check{f} instead of f . The scalar functions are re-defined in such a form that the "sub-integral" formula (34) under the distortion of Ricci tensor (29) is re-written in terms of geometric objects derived for the canonical d-connection,

$$({}_iR + |\bar{\nabla}f|^2)e^{-f} = ({}^F_sR + |{}^F\mathbf{D}\check{f}|^2)e^{-\check{f}} + \Phi.$$

For the second functional, $\mathcal{D} = (h\mathcal{D}, v\mathcal{D})$, we re-scale $\tau \rightarrow \check{\tau}$ and write

$$[\tau({}_iR + |\bar{\nabla}f|^2) + f - 2m] \mu = [\check{\tau}({}^s\hat{\mathbf{R}} + |h\mathcal{D}\check{f}| + |v\mathcal{D}\check{f}|)^2 + \check{f} - 2m] \check{\mu} + \Phi_1,$$

for some Φ and Φ_1 for which $\int_{\overline{V}} \Phi dV = 0$ and $\int_{\overline{V}} \Phi_1 dV = 0$. This provides a proof for⁷

Lemma 4.1 *Considering distortion relations for scalar curvature and Ricci tensor determined by $\overline{\nabla} = \widehat{\nabla} + \widehat{\mathbf{Z}}$ (25), the Perelman's functionals (34) are defined equivalently in N -adapted variables $(L : \widetilde{\mathbf{g}}, \widehat{\mathcal{D}})$,*

$$\mathcal{F}(\widetilde{\mathbf{g}}, \widehat{\mathcal{D}}, \check{f}) = \int_{\overline{V}} ({}^s\widehat{\mathbf{R}} + |h\mathcal{D}\check{f}|^2 + |v\mathcal{D}\check{f}|^2) e^{-\check{f}} dV, \quad (35)$$

$$\mathcal{W}(\widetilde{\mathbf{g}}, \widehat{\mathcal{D}}, \check{f}, \check{\tau}) = \int_{\overline{V}} [\check{\tau} ({}^s\widehat{\mathbf{R}} + |h\mathcal{D}\check{f}|^2 + |v\mathcal{D}\check{f}|^2) + \check{f} - 2m] \check{\mu} dV, \quad (36)$$

where the new scaling function \check{f} satisfies $\int_{\overline{V}} \check{\mu} dV = 1$ for $\check{\mu} = (4\pi\check{\tau})^{-m} e^{-\check{f}}$ and $\check{\tau} > 0$.

In this section, we omit details and proofs which are straightforward consequences of those presented in [3, 4, 5, 6]. For our constructions, we consider operators defined by L via $\overline{\nabla}$ on $\mathcal{T}^E \mathbf{E}$. Using distortions to $\widehat{\mathcal{D}}$ with $\widehat{\mathbf{Z}}$ completely defined by $\widetilde{\mathbf{g}}$, we can study Lagrange–Ricci flows on prolongation Lie algebroids as canonical nonholonomic deformations of Riemannian evolution on associated vector/tangent bundles.

We can construct the canonical Laplacian operator, $\widehat{\Delta} := \widehat{\mathcal{D}} \widehat{\mathcal{D}}$ determined by the canonical d-connection $\widehat{\mathcal{D}}$, a "standard" Laplace operator $\overline{\Delta} = \overline{\nabla} \overline{\nabla}$, and consider parameter $\tau(\chi)$, $\partial\tau/\partial\chi = -1$. For simplicity, we shall not include the normalized term. The distortion (25) results in

$$\begin{aligned} \Delta &= \widehat{\Delta} + {}^Z\widehat{\Delta}, \quad {}^Z\widehat{\Delta} = \widehat{\mathbf{Z}}_{\overline{\alpha}} \widehat{\mathbf{Z}}^{\overline{\alpha}} + [\widehat{\mathbf{D}}_{\overline{\alpha}} (\widehat{\mathbf{Z}}^{\overline{\alpha}}) + \widehat{\mathbf{Z}}_{\overline{\alpha}} \widehat{\mathbf{D}}^{\overline{\alpha}}]; \\ \overline{R}_{\overline{\beta}\overline{\gamma}} &= \widehat{\mathbf{R}}_{\overline{\beta}\overline{\gamma}} + \widehat{\mathbf{Z}} ic_{\overline{\beta}\overline{\gamma}}, \quad {}_sR = {}_s\widehat{\mathbf{R}} + \widetilde{\mathbf{g}}^{\overline{\beta}\overline{\gamma}} \widehat{\mathbf{Z}} ic_{\overline{\beta}\overline{\gamma}} = {}_s\widehat{\mathbf{R}} + {}_s\widehat{\mathbf{Z}}, \\ {}_s\widehat{\mathbf{Z}} &= \widetilde{\mathbf{g}}^{\overline{\beta}\overline{\gamma}} \widehat{\mathbf{Z}} ic_{\overline{\beta}\overline{\gamma}} = {}_h\widehat{\mathbf{Z}} + {}_v\widehat{\mathbf{Z}}, \quad {}_h\widehat{\mathbf{Z}} = \widetilde{\mathbf{g}}^{\alpha\beta} \widehat{\mathbf{Z}} ic_{\alpha\beta}, \quad {}^F_v\widehat{\mathbf{Z}} = \widetilde{\mathbf{g}}^{AB} \widehat{\mathbf{Z}} ic_{AB}; \\ {}_s\overline{R} &= {}_h\overline{R} + {}_v\overline{R}, \quad {}_h\overline{R} := \widetilde{\mathbf{g}}^{\alpha\beta} \overline{R}_{\alpha\beta}, \quad {}_v\overline{R} = \widetilde{\mathbf{g}}^{AB} \overline{R}_{AB}, \end{aligned} \quad (37)$$

where, for convenience, capital indices $A, B, C \dots$ are for distinguishing v -components even the prolongation Lie algebroid is constructed for $\mathbf{P} = \mathbf{E}$. Using such deformations and a proof similar to that in Proposition 1.5.3 of [6], we obtain

⁷similar N -adapted constructions were considered in Claim 3.1 in Ref. [9, 10] for nonholonomic manifolds and Lagrange spaces on TM

Theorem 4.1 *The Lagrange–Ricci flows for $\widehat{\mathcal{D}}$ preserving a symmetric metric structure $\widetilde{\mathbf{g}}$ and Lie algebroid structure for prolonated $\mathcal{T}^E\mathbf{E}$ can be characterized by this system of geometric flow equations:*

$$\begin{aligned}\frac{\partial \widetilde{\mathbf{g}}_{\alpha\beta}}{\partial \chi} &= -2 \left(\widehat{\mathbf{R}}_{\alpha\beta} + \widehat{\mathbf{Z}} ic_{\alpha\beta} \right), \quad \frac{\partial \widetilde{\mathbf{g}}_{AB}}{\partial \chi} = -2 \left(\widehat{\mathbf{R}}_{AB} + \widehat{\mathbf{Z}} ic_{AB} \right), \\ \widehat{\mathbf{R}}_{\alpha A} &= -\widehat{\mathbf{Z}} ic_{\alpha A}, \quad \widehat{\mathbf{R}}_{A\alpha} = \widehat{\mathbf{Z}} ic_{A\alpha}, \\ \frac{\partial \widehat{f}}{\partial \chi} &= - \left(\widehat{\Delta} + {}^Z \widehat{\Delta} \right) \widehat{f} + \left| \left(\widehat{\mathbf{D}} + \widehat{\mathbf{Z}} \right) \widehat{f} \right|^2 - {}_s \widehat{\mathbf{R}} - {}_s \widehat{\mathbf{Z}},\end{aligned}\tag{38}$$

and the property that

$$\begin{aligned}\frac{\partial}{\partial \chi} \mathcal{F}(\widetilde{\mathbf{g}}, \widehat{\mathcal{D}}, \widehat{f}) &= \int_{\widehat{\mathcal{V}}} [|\widehat{\mathbf{R}}_{\alpha\beta} + \widehat{\mathbf{Z}} ic_{\alpha\beta} + (\widehat{\mathbf{D}}_{\alpha} + \widehat{\mathbf{Z}}_{\alpha})(\widehat{\mathbf{D}}_{\beta} + \widehat{\mathbf{Z}}_{\beta})\widehat{f}|^2 + \\ &|\widehat{\mathbf{R}}_{AB} + \widehat{\mathbf{Z}} ic_{AB} + (\widehat{\mathbf{D}}_A + \widehat{\mathbf{Z}}_A)(\widehat{\mathbf{D}}_B + \widehat{\mathbf{Z}}_B)\widehat{f}|^2] e^{-\widehat{f}} dV, \quad \int_{\widehat{\mathcal{V}}} e^{-\widehat{f}} dV = \text{const.}\end{aligned}$$

Proof. For distortions (37), we can redefine the scaling functions from above Lemma in different form. Similarly to [9, 10] we can construct on $\mathcal{T}^E\mathbf{E}$ the corresponding system of Ricci flow evolution equations for $\widehat{\mathcal{D}}$,

$$\begin{aligned}\frac{\partial \widetilde{\mathbf{g}}_{\alpha\beta}}{\partial \chi} &= -2\widehat{\mathbf{R}}_{\alpha\beta}, \quad \frac{\partial \widetilde{\mathbf{g}}_{AB}}{\partial \chi} = -2\widehat{\mathbf{R}}_{AB}, \\ \frac{\partial \widehat{f}}{\partial \chi} &= -\widehat{\Delta} \widehat{f} + \left| \widehat{\mathcal{D}} \widehat{f} \right|^2 - {}_h \widehat{R} - {}_v \widehat{R},\end{aligned}\tag{39}$$

which can be from the functional $\widehat{\mathcal{F}}(\widetilde{\mathbf{g}}, \widehat{\mathcal{D}}, \widehat{f}) = \int_{\widehat{\mathcal{V}}} ({}_s \widehat{R} + |\widehat{\mathcal{D}} \widehat{f}|^2) e^{-\widehat{f}} dV$. The conditions $\widehat{R}_{\alpha A} = 0$ and $\widehat{R}_{A\alpha} = 0$ must be imposed in order to model evolution only with symmetric metrics. \square

We note that under Ricci flows the N-adapted frames also depend on parameter χ following certain evolution formulas. For TM , such a Corollary is proven in Ref. [10]. Re-defining indices for $\mathcal{T}^E\mathbf{E}$, those formulas can be used for flow evolution of frames of type (15) and (16).

Finally, we discuss the statistical model which can be elaborated for Ricci flows of mechanical systems. By definition, the functional ${}_1\mathcal{W}$ is analogous to minus entropy [3] and this property was proven for metric compatible nonholonomic and Lagrange–Finsler Ricci flows [9, 10] with functionals $\widehat{\mathcal{W}}$ written for $\widehat{\mathbf{D}}$. Similar constructions can be performed on $\mathcal{T}^E\mathbf{E}$.

Let us consider a partition function $Z = \int \exp(-\beta E) d\omega(E)$ for the canonical ensemble at temperature β^{-1} being defined by the measure taken

to be the density of states $\omega(E)$. The thermodynamical values are computed for average energy, $\langle E \rangle := -\partial \log Z / \partial \beta$, entropy $S := \beta \langle E \rangle + \log Z$ and fluctuation $\sigma := \langle (E - \langle E \rangle)^2 \rangle = \partial^2 \log Z / \partial \beta^2$.

Theorem 4.2 *Any family of Lagrangians under Ricci evolution on $\mathcal{T}^E \mathbf{E}$ is characterized by thermodynamic values*

$$\begin{aligned} \langle \tilde{E} \rangle &= -\tilde{\tau}^2 \int_{\mathcal{V}} (\hat{s}R + |\hat{\mathcal{D}}\tilde{f}|^2 - \frac{m}{\tilde{\tau}}) \tilde{\mu} dV, \\ \tilde{S} &= - \int_{\mathcal{V}} [\tilde{\tau}(\hat{s}R + |\hat{\mathcal{D}}\tilde{f}|^2) + \tilde{f} - 2m] \tilde{\mu} dV, \\ \tilde{\sigma} &= 2 \tilde{\tau}^4 \int_{\mathcal{V}} [|\hat{\mathbf{R}}_{\alpha\bar{\beta}} + \hat{\mathcal{D}}_{\alpha}\hat{\mathcal{D}}_{\bar{\beta}}\tilde{f} - \frac{1}{2\tilde{\tau}}\tilde{\mathbf{g}}_{\alpha\bar{\beta}}|^2] \tilde{\mu} dV. \end{aligned}$$

Proof. Similar computations, in not N-adapted, or N-adapted forms, are given in [6, 9, 10]. On prolongation Lie algebroids, we have to use the partition function $\tilde{Z} = \exp \left\{ \int_{\mathcal{V}} [-\tilde{f} + m] \tilde{\mu} dV \right\}$. \square

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